Certified surface approximations using the interval Krawczyk test

Joint work with Michael Burr and Jonathan D. Hauenstein

Kisun Lee (Clemson University) - kisunl@clemson.edu

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Certified algebraic variety approximation

Given: a smooth variety $Z = \mathscr{V}(f_1, \ldots, f_{n-d})$,
an (approximate) point $\hat{z}$ on Z , and
a compact region R that contains $\hat{z}$.

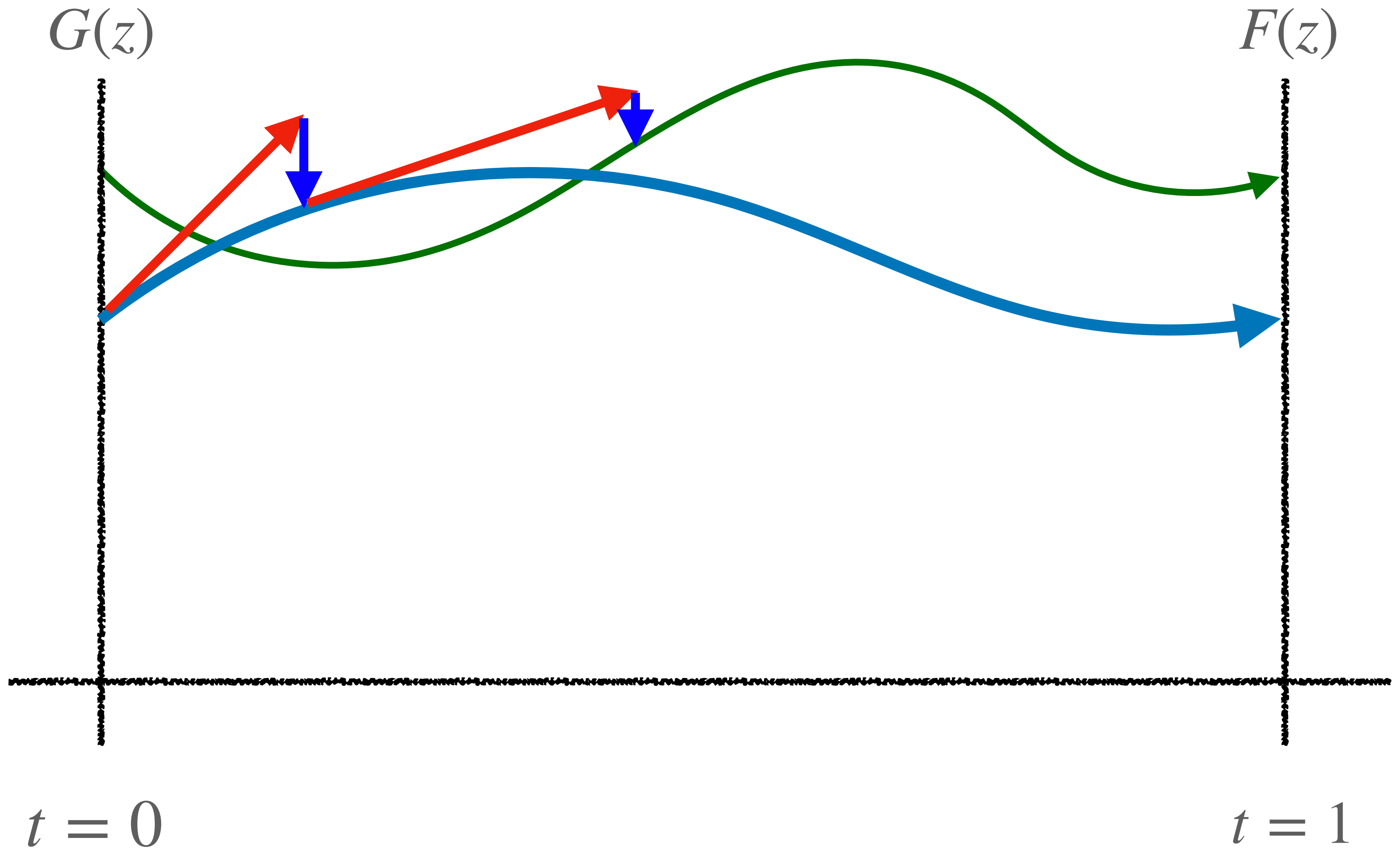
Starting at $\hat{z}$, **track** along the variety Z in a **certified** manner to construct an isotopic approximation of Z.

Possible stopping criteria:

- Track until the computed part hits the boundary of R,
- conclude that the variety Z is compact.

Motivations

Certified homotopy tracking



Homotopy tracking finds solutions to a polynomial system by tracking homotopy

$$H(t, z) = (1 - t)\gamma G(z) + tF(z), \quad t \in [0,1].$$

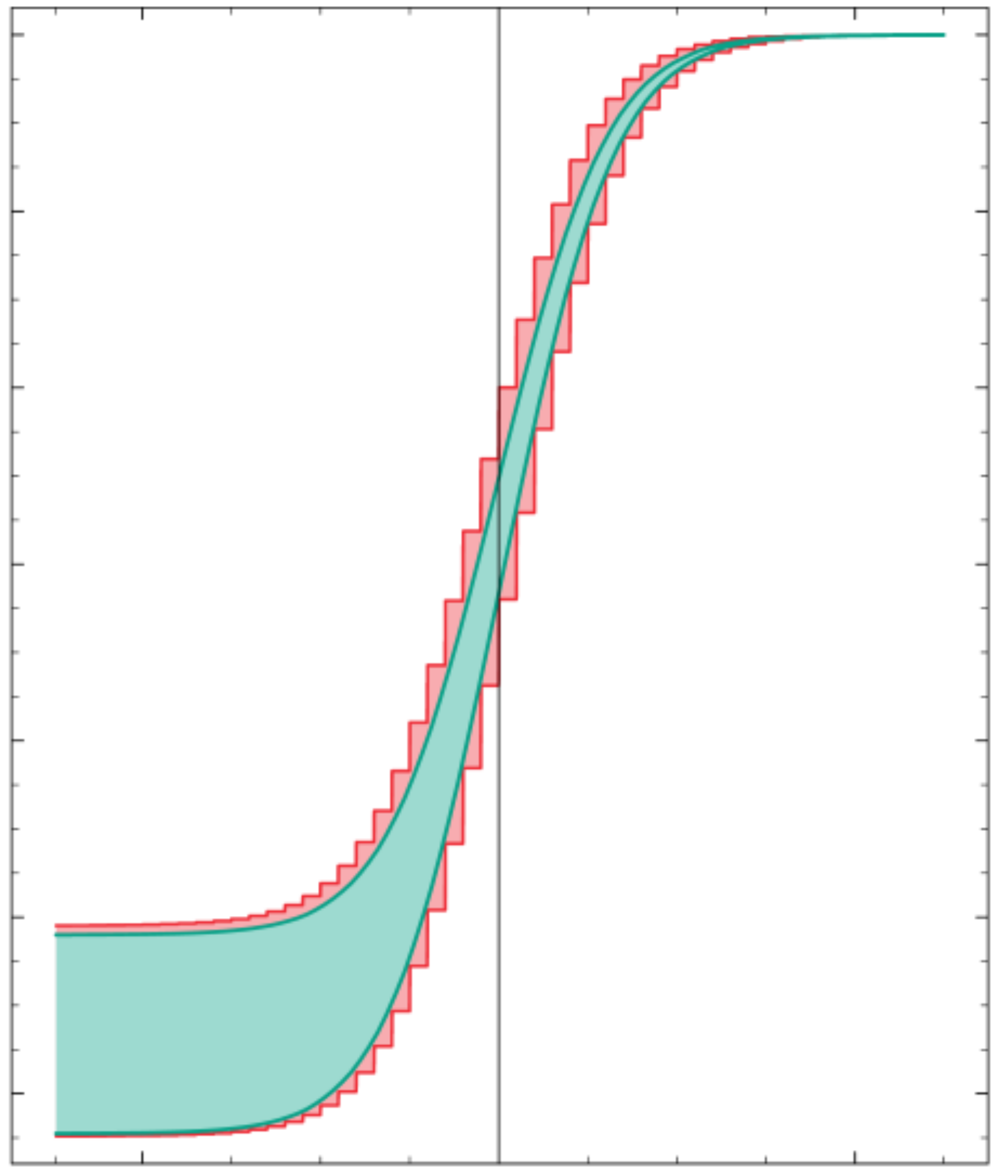
Solve F (target system) by constructing a homotopy with G (start system) whose solutions are known.

A path-jumping may occur when there is a nearby path.

Certified homotopy tracking guarantees that the solution path is tracked without jumping.

(Beltran-Leykin 2012 2013, Martin-Goldsztejn-Granvilliers-Jermann 2013, Hauenstein-Haywood-Liddell 2014, van der Hoeven 2015, Burr-Yap-Xu 2018)

Krawczyk method



Interval arithmetic executes reliable computation, by bounding the rounding error

- Addition:

$$[x_1, x_2] + [y_1, y_2] = [x_1 + y_1, x_2 + y_2]$$

- Multiplication:

$$[x_1, x_2] \cdot [y_1, y_2]$$
$$= [\min\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}, \max\{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}]$$

Certified tracking

Krawczyk method

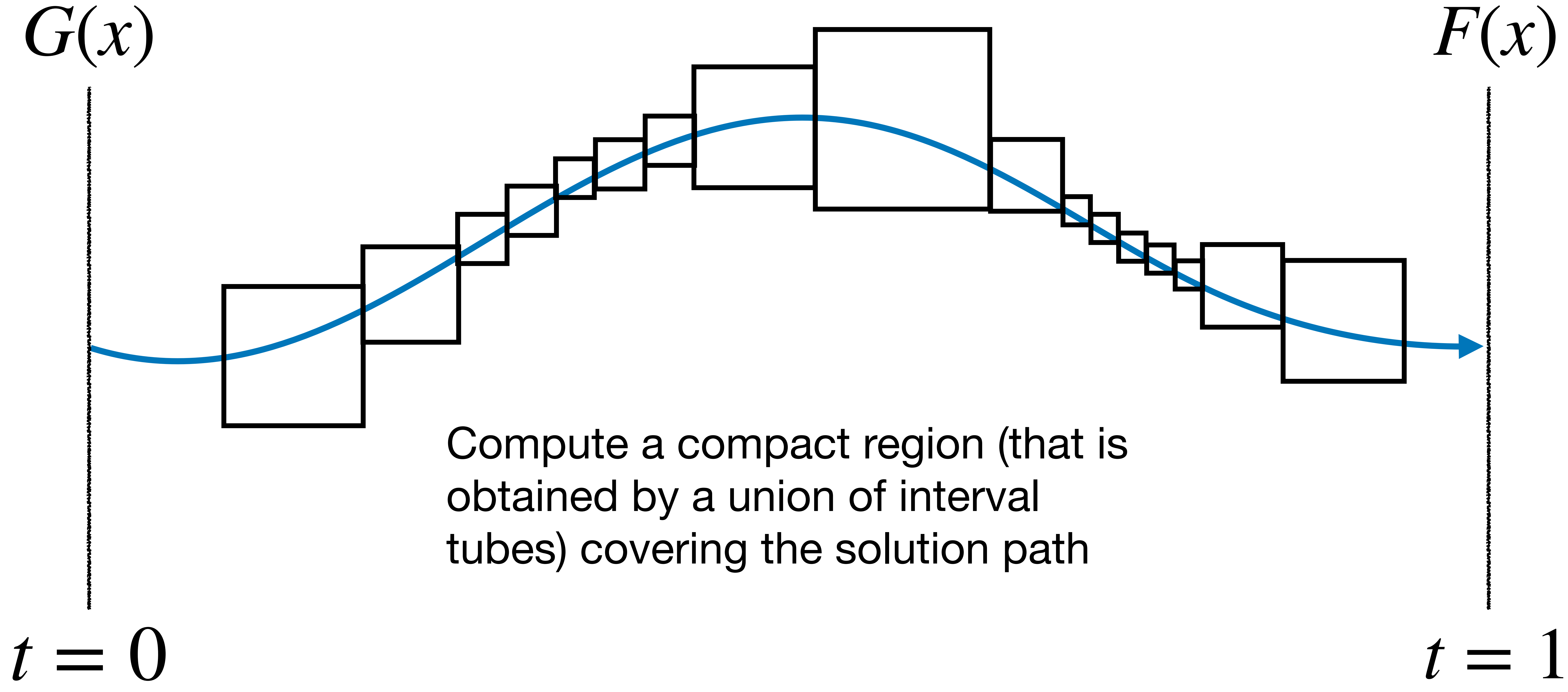
The Krawczyk method combines interval arithmetic and Newton's method. For a system $F:\mathbb{C}^n \to \mathbb{C}^n$, an interval box I with the midpoint x, and an $n \times n$ invertible matrix A, define the **Krawczyk operator**

$$-AF(x)+\big(Id - A\,\square\,J_F(I)\big)(I-x)$$

Theorem (Krawczyk 1969). For a given interval box I with the radius r and a constant $\rho \in (0,1)$, if

$$-AF(x)+\big(Id - A\,\square\,J_F(I)\big)(I-x) \subset \rho I,$$

Then, there is a unique solution $x^\star$ to F in I.
Also, Newton iteration applied at a point in I converges to $x^\star$.
In this case, we say that x is a **ρ-approximate solution** to F.
ρ works as a refinement parameter (smaller $\rho \Rightarrow$ tighter box)


$G(x)$
$F(x)$
Compute a compact region (that is
obtained by a union of interval
tubes) covering the solution path
$t = 0$
$t = 1$

Certified tracking

Two simultaneous independent works

ISSAC 2024

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Certified homotopy tracking using the Krawczyk method

Authors: Timothy Duff, Kisun Lee | Authors Info & Claims

ISSAC '24: Proceedings of the 2024 International Symposium on Symbolic and Algebraic Computation • Pages 274 - 282
https://doi.org/10.1145/3666000.3669699

Published: 16 July 2024 Publication History

Check for updates

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Validated Numerics for Algebraic Path Tracking

Authors: Alexandre Guillemot, Pierre Lairez | Authors Info & Claims

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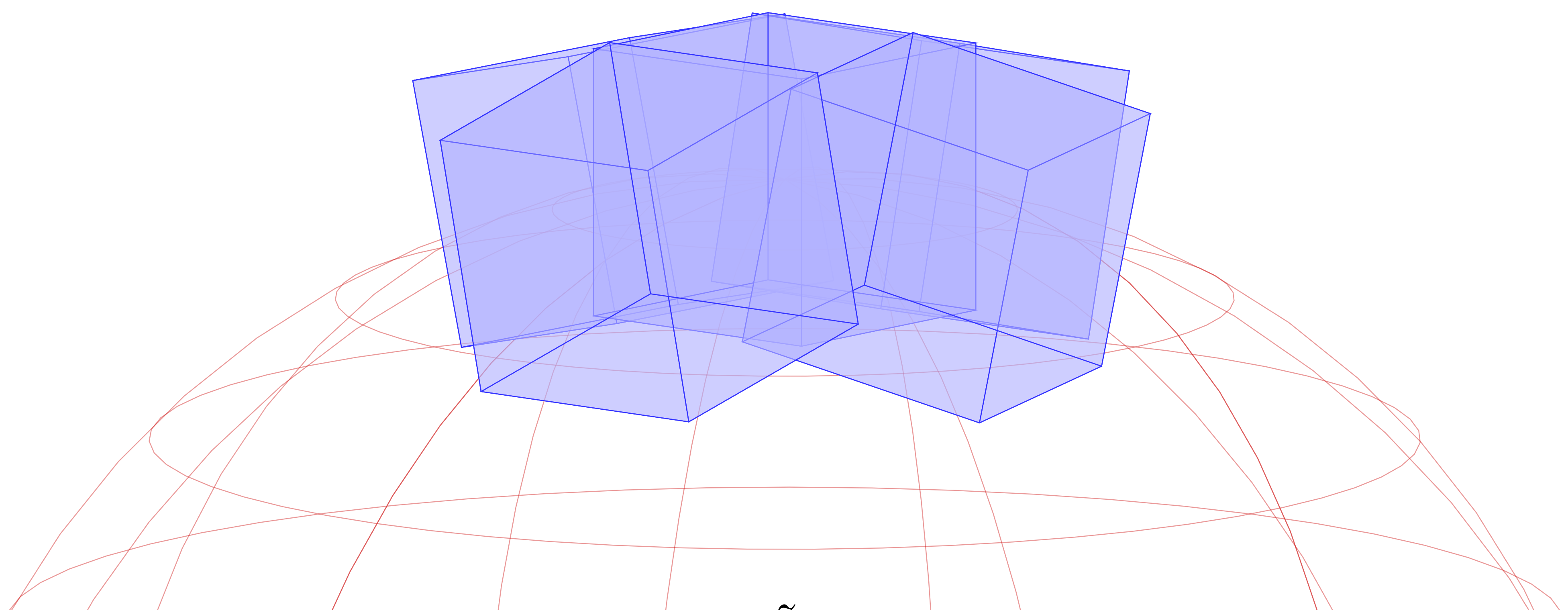
Interval Krawczyk test



Certified homotopy tracking relies on the fact that every slice of the box intersects the solution path at exactly one point.

The **interval Krawczyk test** generalizes this to an arbitrary algebraic variety.

For $\hat{z} \in \mathbb{R}^n$ (near $\mathscr{V}(F)$) with $\hat{z} = (\pi_d(\hat{z}), \pi_{-d}(\hat{z}))$,

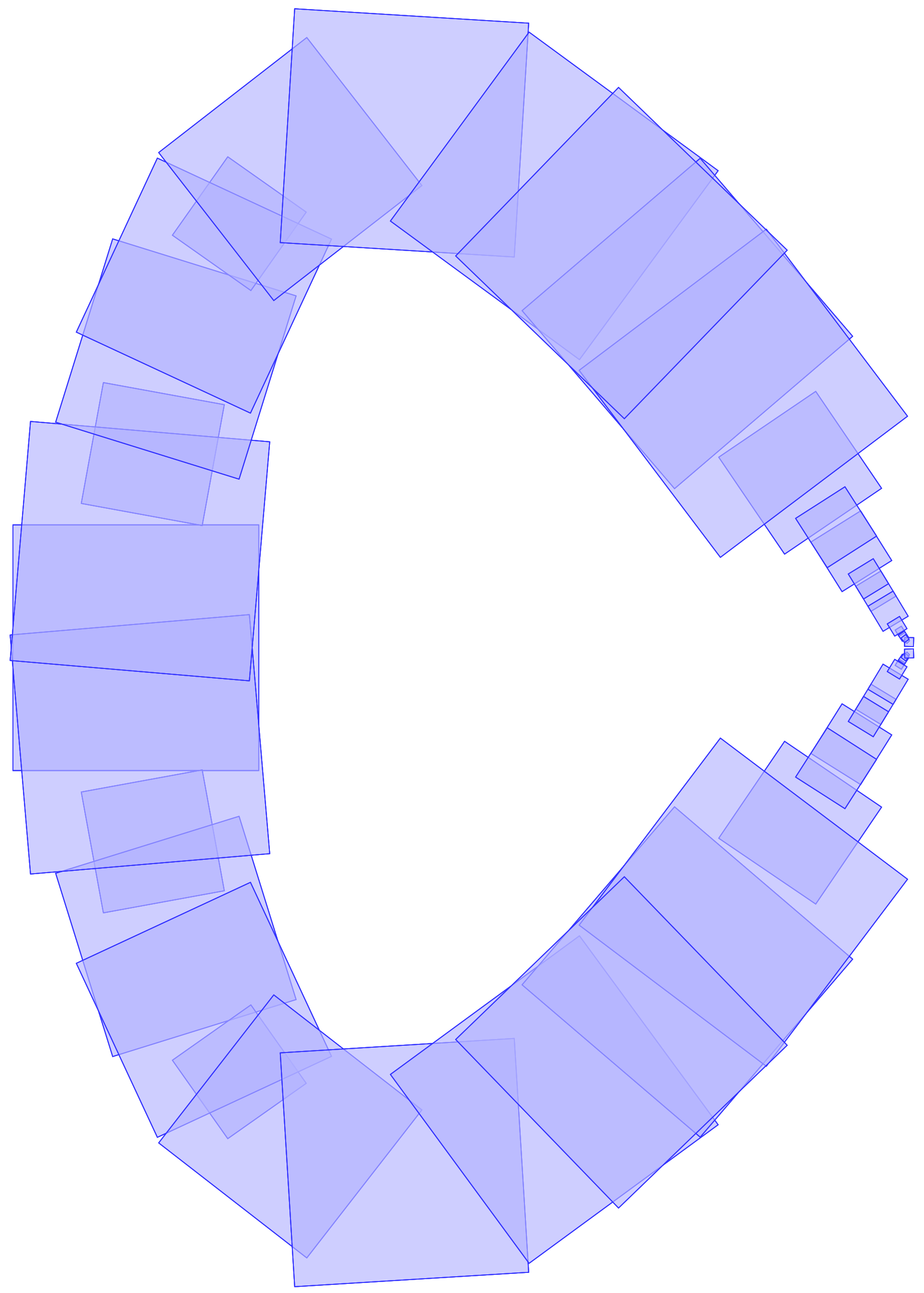
$$-A \square F(I, \pi_{-d}(\hat{z})) + (Id_{n-d} - A \square J_F(I, J))(J - \pi_{-d}(\hat{z}))$$

where $I := \pi_d(\hat{z}) + r_1[-1,1]^d$ and $J := \pi_{-d}(\hat{z}) + r_2[-1,1]^{n-d}$

The interval Krawczyk test certifies that every fiber intersects the variety exactly once.

When the interval Krawczyk test passes, we set $\hat{z}$ by the center of the sides $\partial I \times J$ and make new certified boxes.

Component test



$\{x^3 - 2.9995x - y^2 + 2, z\}$ with $\rho = \dfrac{7}{8}$

Whenever two boxes intersect, we perform the component test.

Inclusion test

- Construct a certified box inside the overlap.
- Success $\Rightarrow$ same sheet

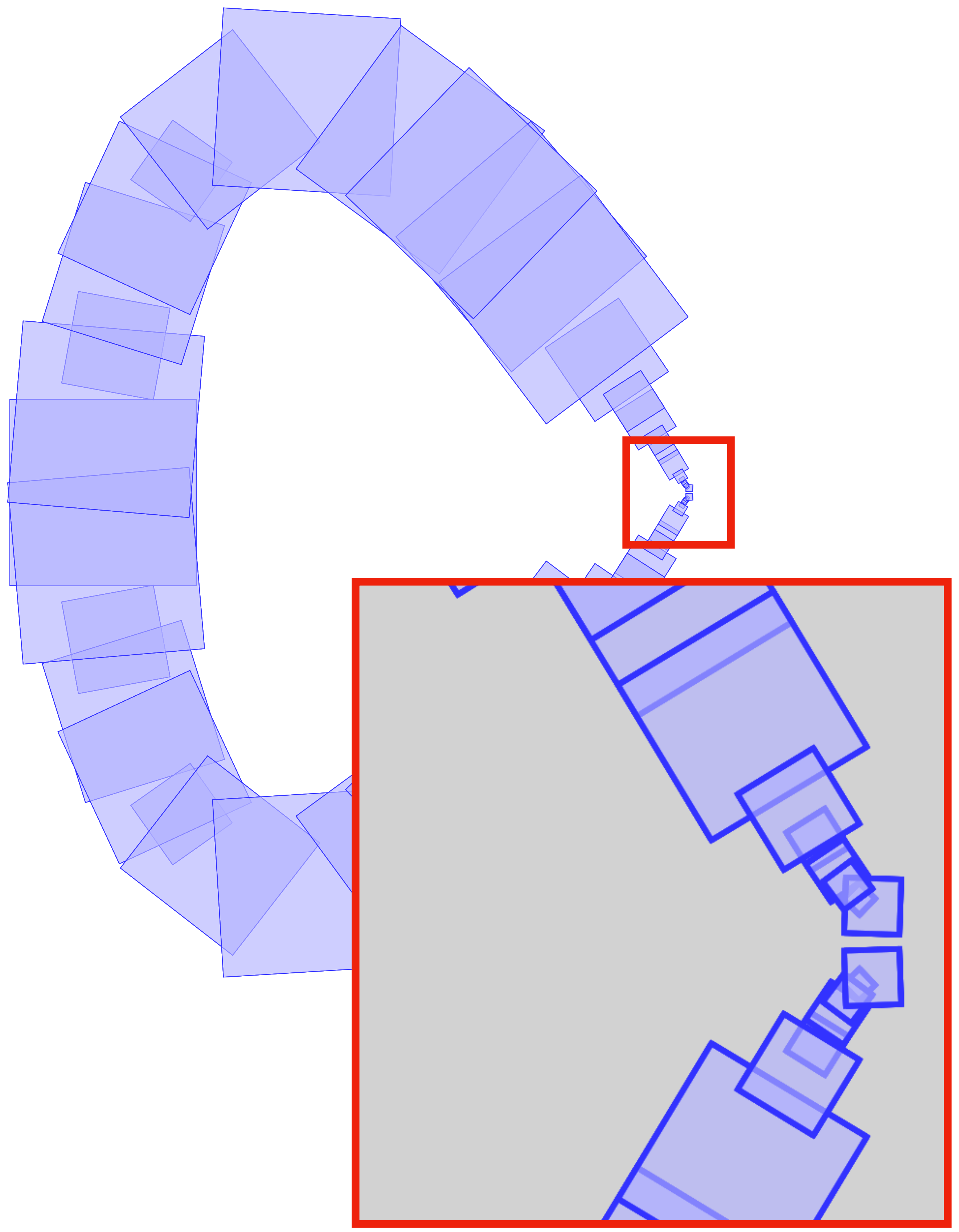
Exclusion test

- Refine the boxes.
- No refined boxes intersect $\Rightarrow$ different sheets

If undecided

- Refine the boxes with smaller ρ.
- Repeat until one test succeeds.

Component test



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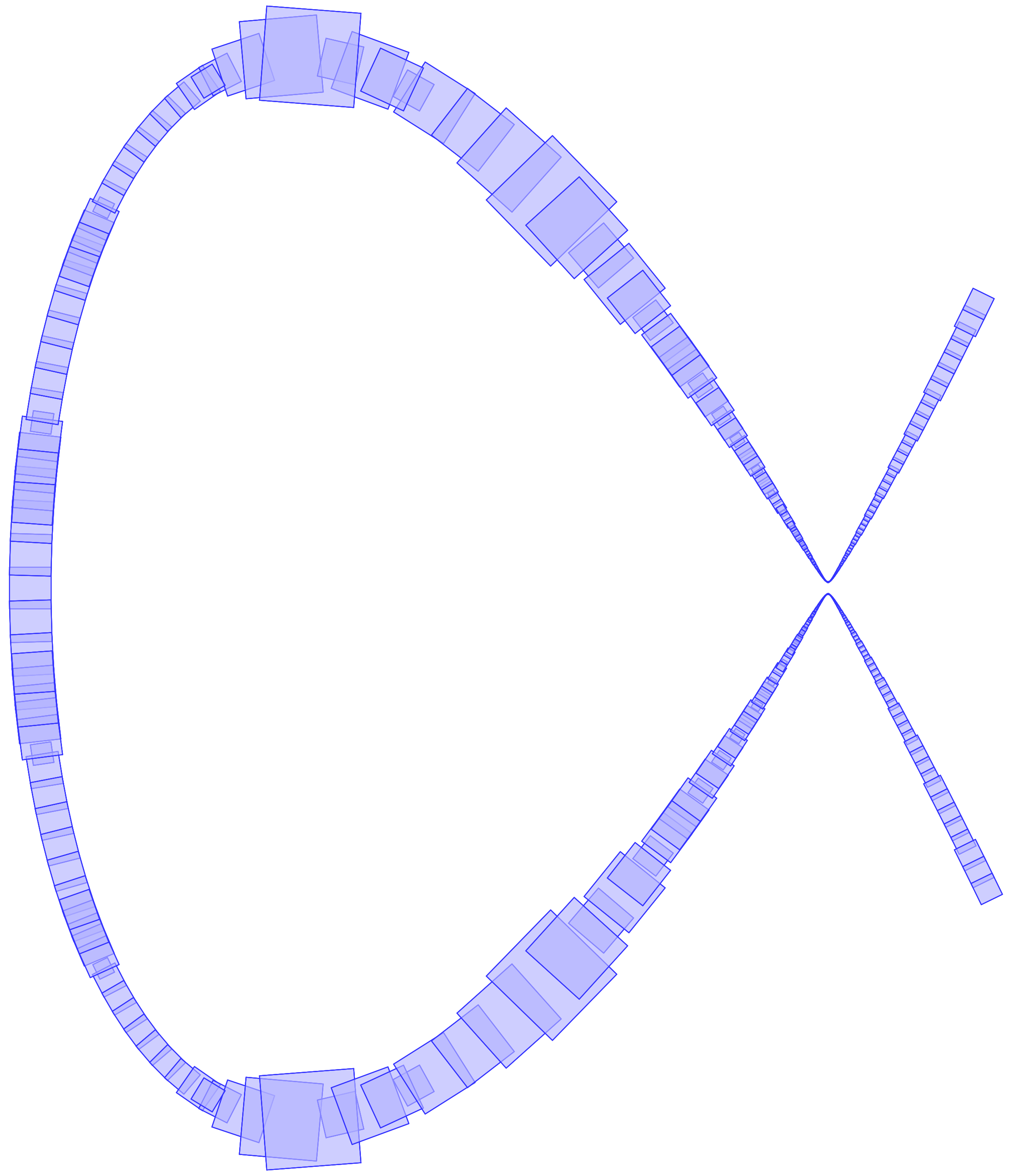

$\{x^3 - 2.9995x - y^2 + 2, z\}$ with $\rho = \frac{1}{8}$

Whenever two boxes intersect, we perform the component test.

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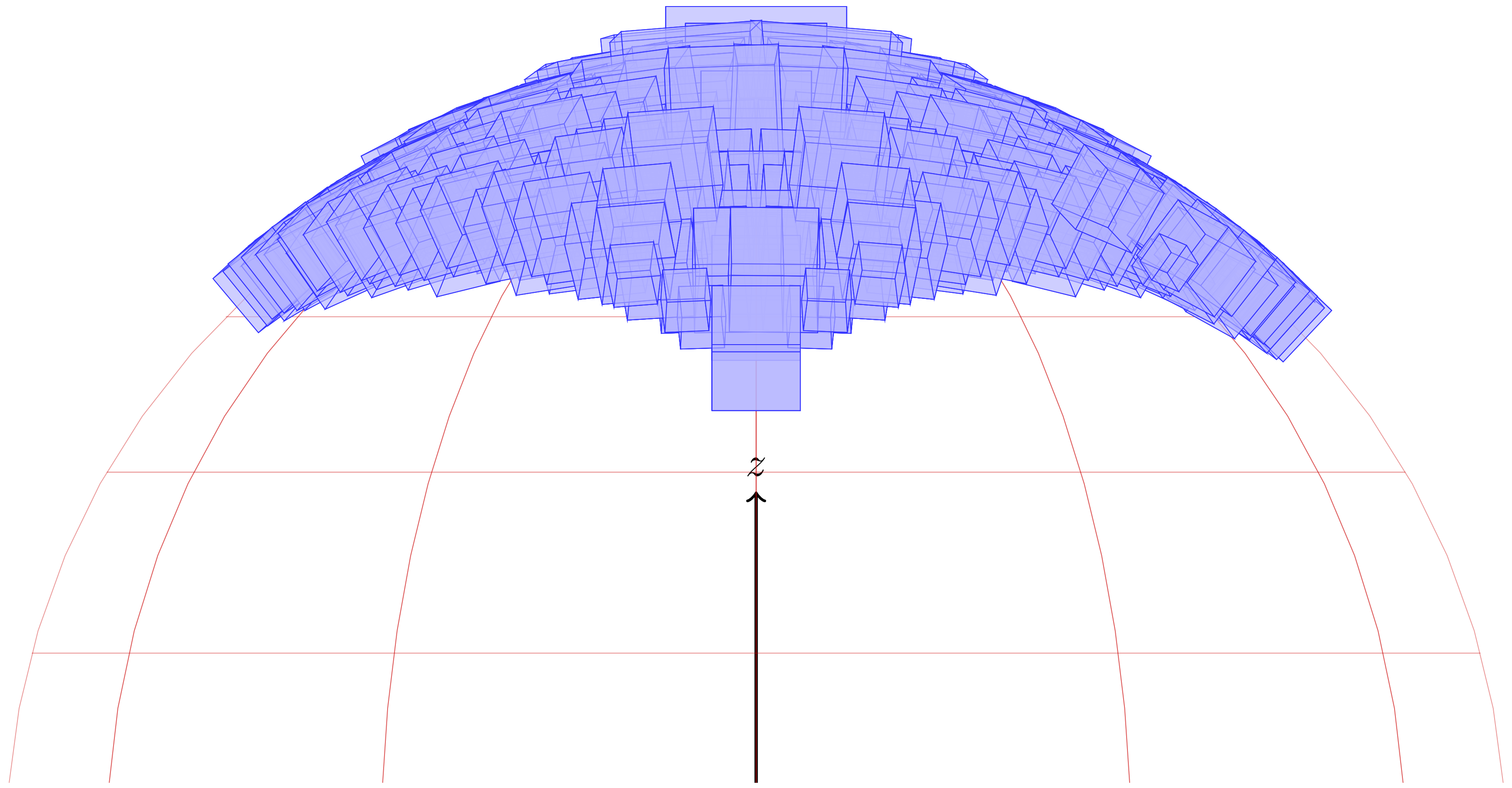
Unitary transformation

Local coordinate change



The interval Krawczyk test applies when the variety is a local graph with respect to the chosen projection.

At every certified point, we rotate the coordinates so that the tangent space becomes nearly horizontal.

$$\tilde{F}(z) = U^* F(Vz), \quad \tilde{z} = V^* z$$

where V is obtained from the SVD of the Jacobian.

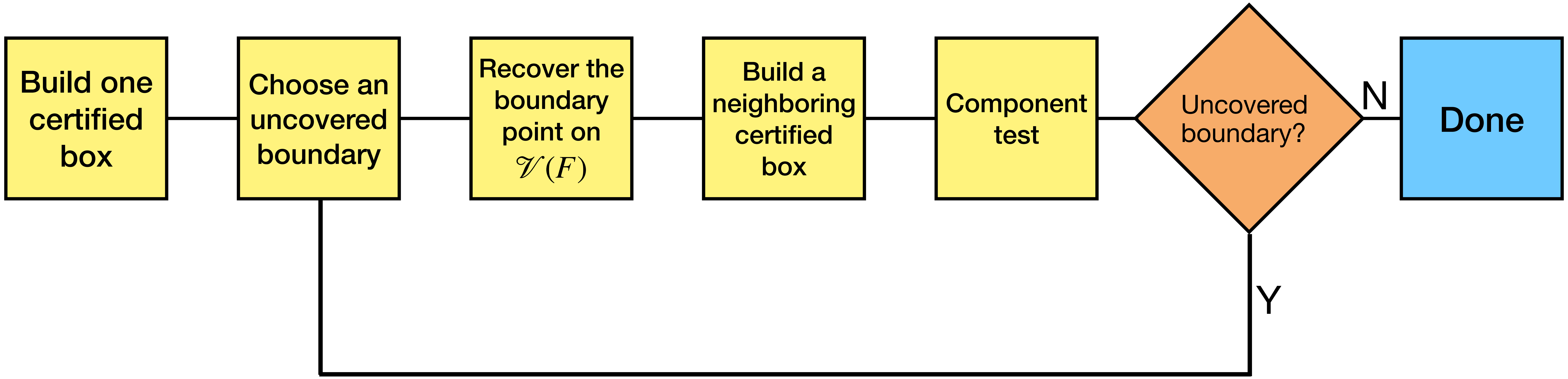
Given: A smooth variety defined by $Z = \mathscr{V}(f_1, \ldots, f_{n-d}) \subset \mathbb{R}^n$,

a given point $\hat{z}$,

an initial radius $r = (r_1, r_2)$,

and a constant ρ

(and a compact region D)



Future works

Remaining challenges

Global correctness

How to check whether the approximation is topologically correct?

- **Via Čech/Rips reconstruction**
  Build a certified nerve of the local patches.
- **Via ellipsoid-based reconstruction**
  Adapt ellipsoid-based deformation retraction to certified surface patches.

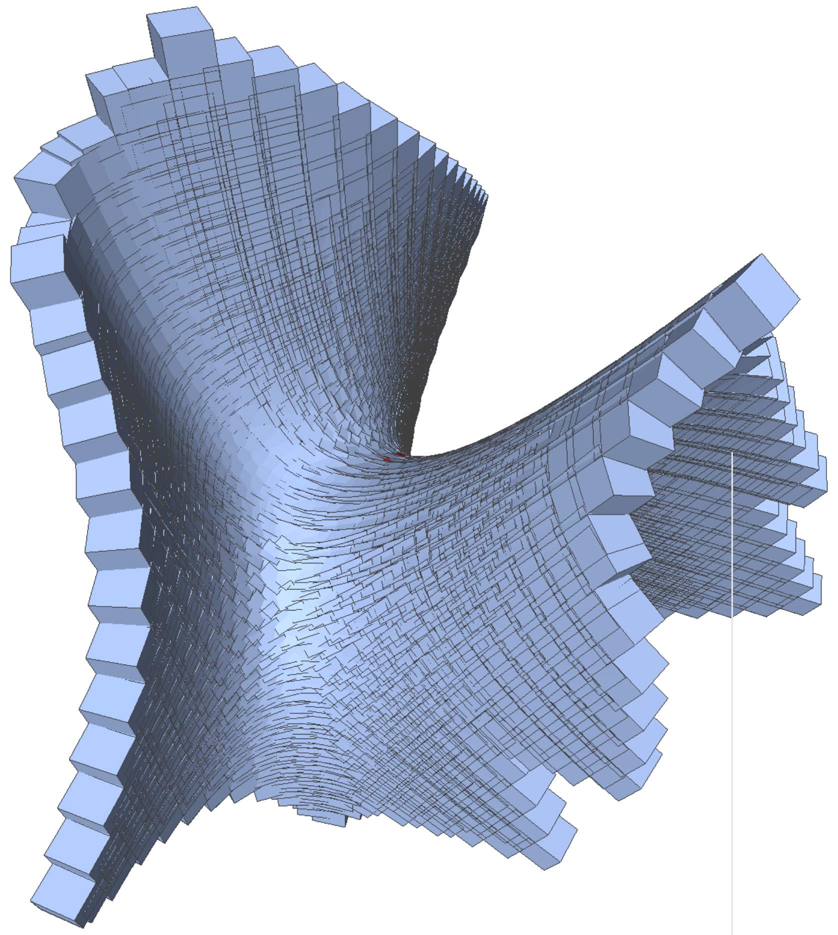
Complexity analysis

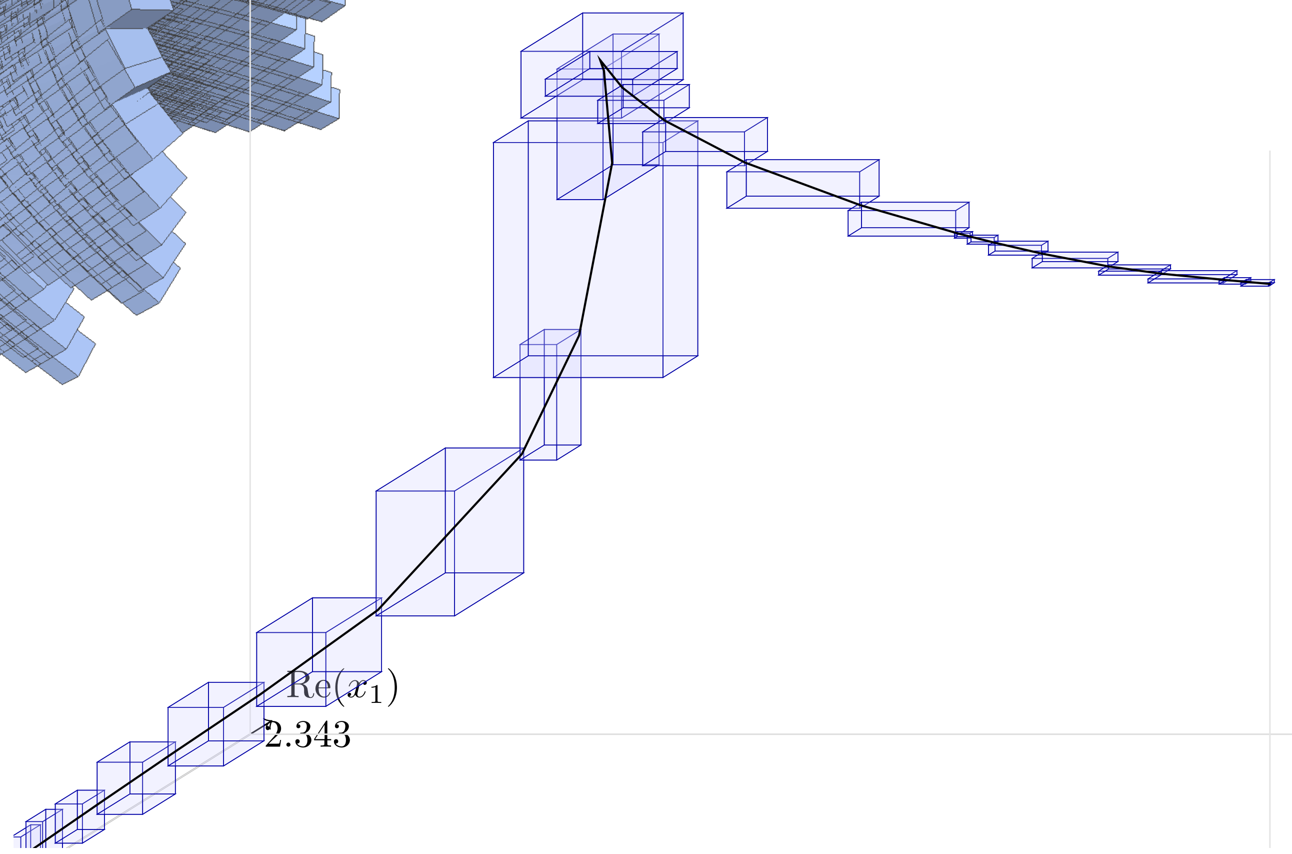
Study the number and sizes of certified patches to the local conditioning and geometry of the variety.

(ongoing work with Burr-Byrd-Tonelli-Cueto)

Closing advertisement

CertifiedHomotopyTracking.jl





CertifiedHomotopyTracking.jl

CertifiedHomotopyTracking.jl

CertifiedHomotopyTracking.jl provides interval-certified tools for tracking homotopy paths and relevant computations (e.g., computing monodromy actions and certified local approximations of algebraic varieties). It pairs Symbolics/Nemo-style model construction with ACB interval arithmetic, with interactions to HomotopyContinuation.jl and GAP.

Main functionalities are like below:

- A Posteriori Certification
- Tracking
- Monodromy
- Variety Approximations

Julia package for certified homotopy-based computations

- Certified homotopy tracking / a posteriori path certification
- Galois/monodromy action computation (Duff-L. 2026+)
- Certified variety approximations (Burr-Hauenstein-L. 2026)

Try `using CertifiedHomotopyTracking` in Julia now!

Origin of the word ‘symposium’

sym·po·si·um

/simˈpōzēəm/

Origin

late 16th century: via Latin from Greek sumposion, from sumpotēs ‘fellow drinker’, from sun- **‘together’** + potēs **‘drinker’**.

Let’s drink together!

Thank you for your attention!